Chapter 3

Foundations of Geometry 1: Points, Lines, Segments, Angles

3.1 An Introduction to Proof

Syllogism: The abstract form is:
1. All $A$ is $B$.
2. $X$ is $A$
3. $\therefore$ $X$ is $B$

Example: Let’s think about an example.
Remark: Syllogism provides the basis for moving from the general to the particular, a process called deductive logic. On the other hand, Inductive logic, moves from the particular to the general.

If-then statements, conditionals

$$p \rightarrow q$$

read: "$p$ implies $q$, or If $p$, then $q$.

The condition $p$ is called the hypothesis, the Given part, and $q$ the conclusion-the prove part.

Negate a statement: if $p$ is any condition, $\sim p$ means not $p$.

Converse, Contrapositive:

<table>
<thead>
<tr>
<th>Given conditional</th>
<th>its converse</th>
<th>its contrapositive</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p \rightarrow q$</td>
<td>$q \rightarrow p$</td>
<td>$\sim q \rightarrow \sim p$</td>
</tr>
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Remark: The contrapositive is logically equivalent to the original conditional, but the converse is not.

Example 3.1 For example: Right angles are congruent. Find the if-then conditional of this statements, and find its converse and contrapositive.
Example 3.2  If two sides of a triangle are congruent, then the angles opposite are congruent.

Converse:

In this case we have two statements are logically equivalent.

Logically equivalent: if we have

\[ p \rightarrow q \quad \text{and} \quad q \rightarrow p \]

i.e.,

\[ p \leftrightarrow q \]

we say that \( p \) and \( q \) are logically equivalent, or we can also say that \( q \) characterizes \( p \).

Direct Proofs
direct proof, which is the embodiment of the propositional syllogism:
1. \( p \) implies \( q \)
2. \( q \) implies \( r \)
3. \( \therefore p \) implies \( r \).

Example 3.3  Given: \( \angle A \) and \( \angle B \) are right angles, Prove: \( \angle A \) is congruent to \( \angle B \).

Proof:

\begin{array}{ll}
\text{Conclusions} & \text{Justifications} \\
(1) \text{The measures of } \angle A \text{ and } \angle B \text{ are each } 90^0. & \text{Definition (of right angle)} \\
(2) m\angle A = m\angle B & \text{Algebra} \\
(3) \therefore \angle A \cong \angle B & \text{Definition of congruence} \\
\end{array}

Indirect Proof: Prove by contradiction

Assume that the conclusion you are trying to prove is false, and then show that this leads to a contradiction of the hypothesis. In another words, to show that \( p \) implies \( q \), one assumes \( p \) and not \( q \), and then proceeds to show that not \( p \) follows, which is a contradiction. This means that we prove the contrapositive of the given proposition instead of the proposition itself.

Example 3.4  Show that a prime number > 2 must be an odd number.

Rule of Elimination

Example 3.5  Trichotomy property of real numbers: For any real numbers \( a \) and \( b \), either \( a < b, a = b, \) or \( a > b \).

For example: if we want to show \( x = 0 \), we can show it is impossible that \( x > 0 \) and \( x < 0 \).
3.2 Axioms, Axiomatic Systems

An axiomatic system always contains statements which are assumed without proof—the axioms. These axioms are chosen
(a) for their convenience and efficiency
(b) for their consistency
and, in some cases, (but not always)
(c) for their plausibility

Undefined terms
Every axiom must, of necessity, contain some terms that have been purposely left without definitions—the undefined terms.
For example, in geometry, the most common undefined terms are “point” and “line.”
In reality, a point is a dot with physical dimension, but ideally in geometry, it has no dimension.
A line is that has length without width.

Models for axiomatic systems
A Model for an axiomatic system is a realization of the axioms in some mathematical setting. All undefined terms are interpreted, and all the axioms are true.

Independence and consistency in axiomatic systems
An axiomatic system must be independent (every axiom is essential, none is a logical consequence of the others) and consistent (freedom from contradictions).
The Euclidean geometry has only one model, namely, three-dimensional coordinate geometry, or the equivalent. If an axiomatic system has only one model, it is called categorical. So do the two non-Euclidean geometries.

Example 3.6 Consider the following axiomatic system:
Undefined terms: member, committee
AXIOMS:
1. Every committee is a collection of at least two members.
2. Every member is on exactly one committee.
Find two distinctly different models for this set of axioms, and discuss how it might be made categorical.
Solutions: Let one model be:
Members: John, Dave, Robert, Mary, Kathy, and Jane
Committee:
A: John and Robert
B: Dave and Jane
C: Kathy and Mary
Another model is:
Member: \{a, b, c, d, e, f, g, h\}
Committees:
A: \{a, b, c, d\}
B: \{e, f\}
C: \{g, h\}
we have found two different models, the system is noncategorical.

Suppose we add the axiom:

3. There exist three committees and six members.

Now we have a categorical system: All models would be similar. Every model consists of three committees, with two members on each committee.

### 3.3 Incidence Axioms for Geometry

**Remark:** axioms used in this book are based on the Hilbert axioms.

**Notations:**

Let $S, T$ be two sets,

1. membership: $x \in S$
2. subset: $S \subseteq T$
3. intersection: $S \cap T$
4. union: $S \cup T$

**Incidence axioms:** axioms govern how points, lines, and planes interact.

**Undefined terms:** points, line, plane, space.

(The latter two indicate axioms for three-dimensional space.)

**Universal set** ($S$: all points in space)

- a line: $\ell$ ($\ell \subseteq S$)
- a plane: $P$:
  1) a line lie in a plane: $\ell \subseteq P$
  2) a line pass through a plane at one point: $\ell \cap P = \{A\}$
  3) a line parallel to a plane: $\ell \cap P = \emptyset$

**Axioms for points, lines and planes**

**Axiom 3.7 (I-1)** Each two distinct points determine a line.

**Remark:**

1. notation for a line: $\overline{AB}$ given two distinct points $A$ and $B$.
2. Since $\overline{AB}$ is a set of points, by definition, we have
   
   $$A \in \overline{AB}, \quad B \in \overline{AB}, \quad \therefore \{A, B\} \subseteq \overline{AB}$$

3. The line is unique.
4. This axiom does not assume the concept that there are infinite number of points on a line, let alone a ‘continuum’ (continuous infinite stream) of points.

**Theorem 3.8** If $C \in \overline{AB}$, $D \in \overline{AB}$ and $C \neq D$, then $\overline{CD} = \overline{AB}$. 
Proof: Since \( C, D \in \overrightarrow{AB} \), so \( \overrightarrow{AB} \) passes \( C, D \) as well. By Axiom I, \( \overrightarrow{CD} = \overrightarrow{AB} \). □

Axiom 3.9 (I-2) Three noncollinear points determine a (unique) plane.

Axiom 3.10 (I-3) If two points lie in a plane, then any line containing those two points lies in that plane.

Axiom 3.11 (I-4) If two distinct planes meet, their intersection is a line.

Axiom 3.12 (I-5) Space consists of at least four noncoplanar points, and contains three noncollinear points. Each plane is a set of points of which at least three are noncollinear, and each line is a set of at least two distinct points.

Models for the axioms of incidence
The model with the smallest number of points has four points, six lines, and four planes.

Construction: Arrange four points \( A, B, C, D \) in ordinary \( xyz \)-space, where \( A \) is the origin, and \( B, C, D \) are at a unit distance from \( A \) on each of the three coordinate axes. This forms a tetrahedron \( ABCD \).

Theorem 3.13 If two distinct lines \( l \) and \( m \) meet, their intersection is a single point. If a line meets a plane and is not contained by that plane, their intersection is a point.

Proof: we prove the first part by contradiction. Assume \( l \cap m \) at two distinct points \( A \neq B \). Then \( A, B \) are on both \( l \) and \( m \). Hence \( l = m \) by Axiom I. This contradicts to the distinctness of \( l \) and \( m \).

The second part can also be proved by contradiction. Assume a line \( l \) does not lie in a plane \( P \), i.e., \( l \not\subseteq P \), and \( A \in l \cap P \). Assume point \( B \neq A \) also is a intersection point, i.e., \( B \in l \cap P \). Thus we have \( \{A, B\} \subseteq P \) and \( \{A, B\} \subseteq l \). By Axiom I, \( AB = l \), and by Axiom 3, \( AB \in P \). This in turn gives \( l \subseteq P \), which is a contradiction. This completes the proof. □

3.4 Distance, Ruler Postulate, Segments, Rays, and Angles

We assume that the distance (or metric) between any two points can be measured. Therefore, following axioms are assumed:
Axiom 3.14 (Metric Axioms) D-1: Each pair of points $A$ and $B$ is associated with a unique real number, called the distance from $A$ to $B$, denoted by $AB$.

D-2 For all points $A$ and $B$, $AB \geq 0$, with equality only when $A = B$.

D-3: For all points $A$ and $B$, $AB = BA$.

Remark:
1.) the distance between two points is not affected by the order in which they occur, i.e., $AB = BA$
2. Triangle inequality (will be proved in Chapter 3): Given three points $A, B, C$, then $AB + BC \geq AC$ the equality holds when $B$ “between” (defined later) $A$ and $C$.

Definition and properties of betweenness

Definition 3.15 (Betweenness) For any three points $A, B$ and $C$, we say that $B$ is between $A$ and $C$, and we write $A - B - C$, iff $A, B$, and $C$ are distinct, collinear points, and $AB + BC = AC$.

Theorem 3.16 If $A - B - C$ then $C - B - A$, and neither $A - C - B$ nor $B - A - C$.

Proof: The first statement is trivial (using definition). Let’s prove $A - C - B$ is not true. Assume $A - C - B$, then $AC + CB = AB$. Since $A - B - C$, $AB + BC = AC$. So we have $AC + CB + BC = AB + BC = AC$, i.e., $2BC = 0$. We then have $B = C$, which is a contradiction. Similarly one can prove the last part of the theorem. □

Definition 3.17 (Ordering of four points on a line) If $A, B, C, D$ are four distinct collinear points, then we write $A - B - C - D$ iff the composite of all four betweenness relations $A - B - C$, $B - C - D$, $A - B - D$ and $A - C - D$ are true.

Theorem 3.18 If $A - B - C$, $B - C - D$, and $A - B - D$ hold, then $A - B - C - D$.

Proof: By definition, only need to show $A - C - D$. □

Example 3.19 Suppose that in a certain metric geometry satisfying Axioms D1-D3, points $A, B, C$ and $D$ are collinear and

$AB = 2$, $AC = 3$, $AD = 4$, $BC = 5$, $BD = 6$, and $CD = 7$. What betweenness relations follow, by definition, among these points? Can these four points be place in some order such that a betweenness relation for all four points holds for them, as in the above definition?

Solutions: Note 1) the geometry given here is not the ordinary Euclidean geometry.

2) Use the definition of betweenness to check the four possibilities: for collections of $\{A, B, C\}$, $\{A, B, D\}$, $\{A, C, D\}$, $\{B, C, D\}$

3) Concludes there is no “quadruple” betweenness.
Segments, Rays, and Angles

The concept of betweenness can be used to define segments and rays, and angles.

**Definition 3.20** Segment $AB$: $\overline{AB} = \{ X : A - X - B, X = A, \text{or} X = B \}$

Ray $AB$: $\overrightarrow{AB} = \{ X : A - X - B, A - B - X, X = A, \text{or} X = B \}$

Angle $ABC$: $\angle ABC = \overrightarrow{BA} \cup \overrightarrow{BC}$ (A, B, and C noncollinear)

One also can give a similar formula to the line $AB$ (although a line is an undefined object in the Euclidean Geometry):

$\overline{AB} = \{ X : X - A - B, A - X - B, A - B - X, X = A, \text{or} X = B \}$

The point $A$ and $B$ are called the end points of segment $\overline{AB}$, and point $A$ is the end point (or origin) of ray $\overrightarrow{AB}$. Point $B$ in $\angle ABC$ is called the vertex of the angle, and rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ its sides.

Extended segments, rays

**Definition 3.21** The extension of segment $\overline{AB}$ is either the ray $\overrightarrow{AB}$ (in the direction $B$), the ray $\overrightarrow{BA}$ (in the direction $A$), or the line $\overline{AB}$ (in both directions). The extension of ray $\overrightarrow{AB}$ is just the line $\overline{AB}$.

**Definition 3.22** A segment without its end points is called an open segment, and a ray without its point (origin) is called an open ray. Any point of a segment (or ray) that is not an end point is referred to as an interior point of that segment (or ray).

Ruler Postulate

**Axiom 3.23** (D4: Ruler Postulate) The points of each line $\ell$ may be assigned to the entire set of real numbers $x, -\infty < x < \infty$, called coordinates, in such a way that

1. each point on $\ell$ is assigned to a unique coordinate;
2. no two points are assigned to the same coordinate;
3. any two points on $\ell$ may be assigned the coordinates zero and a positive real number, respectively;
4. if points $A$ and $B$ on $\ell$ have coordinates $a$ and $b$, then $AB = |a - b|$.

ordering of the geometric points on a line and the ordering of their coordinates

**Theorem 3.24** For any line $\ell$ and any coordinate system under the Ruler Postulate, if $A[a], B[b], \text{and} C[c]$ are three points on line $\ell$, with their coordinates, then $A - B - C$ iff either $a < b < c \text{ or } c < b < a$. 
Remark: The Ruler Postulate imposes on each line in our geometry or ray a continuum of points, like the real number line.

Example 3.25 Prove that if \( A - B - C \) holds, then the two rays from \( B \) passing through \( A \) and \( C \) make a line, that is, prove that \( \overrightarrow{BA} \cup \overrightarrow{BC} = \overrightarrow{AC} \).

Proof: It is necessary to show each side of the equality is a subset of the other.

(a) first we show that \( \overrightarrow{BA} \cup \overrightarrow{BC} \subseteq \overrightarrow{AC} \). Let \( X \in \overrightarrow{BA} \), since \( A - B - C \), \( X \in AC \), so \( \overrightarrow{BA} \subseteq \overrightarrow{AC} \); Similarly one can prove \( \overrightarrow{BC} \subseteq \overrightarrow{AC} \). Hence \( \overrightarrow{BA} \cup \overrightarrow{BC} \subseteq \overrightarrow{AC} \).

(b) we next show the reverse inclusion \( \overrightarrow{AC} \subseteq \overrightarrow{BA} \cup \overrightarrow{BC} \). Let \( X \in \overrightarrow{AC} \). There are seven possible cases as related to the given betweenness \( A - B - C \). But in each case it follows that either \( X \in \overrightarrow{BA} \) or \( X \in \overrightarrow{BC} \). Hence \( X \in \overrightarrow{BA} \cup \overrightarrow{BC} \). This completes the proof. \( \square \)

Question: If \( A - B - C \), then is \( \angle BAD = \angle CAB \) ? if they are equal, we need to show \( \overrightarrow{AB} = \overrightarrow{AC} \).

Theorem 3.26 If \( C \in \overrightarrow{AB} \) and \( A \neq C \), then \( \overrightarrow{AB} = \overrightarrow{AC} \).

Proof: Let a coordinate system be determined for line \( \ell = \overrightarrow{AB} \), with coordinates given by \( A[0] \), and \( B[b] \), \( b > 0 \), as guaranteed by the Ruler Postulate.

1) Let \( P[x] \in \overrightarrow{AB} \), then we must have \( x \geq 0 \) by definition of a ray and the Ruler Postulate.

2) Similarly, since \( C[c] \) is on \( \overrightarrow{AB} \), and \( C \neq A \), then \( c > 0 \). Use the same reasoning, A point \( Q[x] \) on \( \overrightarrow{AC} \) must have \( x \geq 0 \).

3) That is, the points of ray \( \overrightarrow{AB} \) and \( \overrightarrow{AC} \) are the same and \( \overrightarrow{AB} = \overrightarrow{AC} \).

\( \square \)

Segment construction Theorem:

Definition 3.27 We define the midpoint of segment \( \overrightarrow{AB} \) to be any point \( M \) on \( \overrightarrow{AB} \) such that \( AM = MB \). Any geometric object passing through \( M \) is said to bisect the segment \( \overrightarrow{AB} \). The measure of a segment \( \overrightarrow{AB} \) is: \( m\overrightarrow{AB} = AB \).

Theorem 3.28 (Segment Construction) If \( \overrightarrow{AB} \) and \( \overrightarrow{CD} \) are two segments and \( AB < CD \), then there exists a unique point \( E \) on ray \( \overrightarrow{CD} \) such that \( AB = CE \), and \( C - E - D \).

Proof: Use Ruler Postulate. Need to prove both existence and uniqueness.

Let a coordinate system determined for the line \( \overrightarrow{CD} \), such that \( C[0] \), and \( D[d] \), we must have \( d = CD \)
Since $AB < CD$, by Ruler Postulate, there exists a unique point $E$ on the line $\overrightarrow{CD}$, such that $E[e]$, and

$$e = AB$$

This shows the uniqueness and existence of $E$.

Now by Ruler Postulate, since $0 < e < d$, we have $C - E - D$.

Corollary 3.29 The midpoint of any segment exists, and is unique.

### 3.5 Angle measure and the Protractor postulate

**Axiom 3.30 (A1: Existence of Angle measure)** Each angle $\angle ABC$ is associated with a unique real number between 0 and 180, called its **measure** and denoted $m\angle ABC$. No angle can have measure 0 or 180.

**Remark**: 1. The degree mark $^o$ (as in $60^0$), is not normally used in the foundations of geometry, because angle measure is just a real number. But, in practice, we use degree signs to indicate the measure is for angles.

2. In axiomatic geometry, it is customary to assume that all angles have measure less than 180. The measure of an angle can not have two values, it only depends on its sides. The concept of *rotation* is not available because our axioms do not include it.

3. It’s possible to allow angles to have measure 180, but this book follows Moise’s foundation for geometry that disallows straight angles.

**Definition 3.31** A point $D$ is an **interior point** of $\angle ABC$ iff there exists a segment $EF$ containing $D$ as an interior point that extends from one side of the angle to the other ($E \in \overrightarrow{BA}$ and $F \in \overrightarrow{BC}$, $E \neq B, F \neq B$).
Remark: this definition is only valid in Euclidean geometry.

**Axiom 3.32 (A2: Angle addition postulate)** If $D$ lies in the interior of $\angle ABC$, then $m\angle ABD + m\angle DBC = m\angle ABC$. Conversely, if $m\angle ABD + m\angle DBC = m\angle ABC$, then ray $\overrightarrow{BD}$ passes through an interior point of $\angle ABC$.

**Definition 3.33 (Betweenness For Rays)** For any three rays $\overrightarrow{BA}$, $\overrightarrow{BD}$ and $\overrightarrow{BC}$ (having the same end point), we say that ray $\overrightarrow{BD}$ lies between rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$, and we write $\overrightarrow{BA}$-$\overrightarrow{BD}$-$\overrightarrow{BC}$, iff the rays are distinct and $m\angle ABD + m\angle DBC = m\angle ABC$.

**Example 3.34** Suppose that the betweenness relation $C$ – $E$ – $D$ – $B$ holds on line $\overrightarrow{BC}$. Show that $\overrightarrow{AC}$-$\overrightarrow{AE}$-$\overrightarrow{AD}$-$\overrightarrow{AB}$ holds (where the definition of betweenness for four rays is directly analogous to that of four points), and that $m\angle CAB = m\angle 1 + m\angle 2 + m\angle 3$. .
Solution: Need to show the betweenness for $\overrightarrow{AC}-\overrightarrow{AE}-\overrightarrow{AD}$, $\overrightarrow{AC}-\overrightarrow{AE}-\overrightarrow{AB}$, $\overrightarrow{AC}-\overrightarrow{AD}-\overrightarrow{AB}$, and $\overrightarrow{AC}-\overrightarrow{AE}-\overrightarrow{AB}$. Since $E$ is interior to $\angle CAD$, $\overrightarrow{AC}-\overrightarrow{AE}-\overrightarrow{AD}$ follows from Axiom A-2. Similarly for the other three betweenness. It follows by definition $\overrightarrow{AC}-\overrightarrow{AE}-\overrightarrow{AD}-\overrightarrow{AB}$.

Now $m\angle CAB = m\angle CAD + m\angle 3 = (m\angle 1 + m\angle 2) + m\angle 3$.

Remark: Similar results hold concurrent rays (having the same end point) and collinear points. We say there is a duality between them.

Axiom 3.35 (A3: Protractor postulate) The set of rays $\overrightarrow{AX}$ lying on one side of a given line $\overrightarrow{AB}$, including ray $\overrightarrow{AB}$, may be assigned to the entire set of real numbers $x$, $0 \leq x < 180$, called coordinates, in such a manner that

1) each ray is assigned to a unique coordinate
2) no two rays are assigned to the same coordinates
3) the coordinate of $\overrightarrow{AB}$ is 0
4) if rays $\overrightarrow{AC}$ and $\overrightarrow{AD}$ have coordinates $c$ and $d$, then $m\angle CAD = |c - d|$.

we state a dual result of Theorem 3.24 next without proof.
if \( e < f < g \), then \( \overrightarrow{AE} - \overrightarrow{AF} - \overrightarrow{AG} \).

### Angle construction Theorem

**Definition 3.36 (Angle Bisector)** We define an angle bisector for \( \angle ABC \) to be any ray \( \overrightarrow{BD} \) lying between the sides \( \overrightarrow{BA} \) and \( \overrightarrow{BC} \) such that \( m\angle ABD = m\angle DBC \).

**Theorem 3.37 (Angle Construction Theorem)** For any two angles \( \angle ABC \) and \( \angle DEF \) such that \( m\angle ABC < m\angle EDF \), there is a unique ray \( \overrightarrow{EG} \) such that \( m\angle ABC = m\angle GEF \) and \( \overrightarrow{Ed} - \overrightarrow{EG} - \overrightarrow{EF} \).

**Corollary 3.38** The bisector of any angle exists and is unique.

**Proof of the theorem** (Outline):
1. Let \( a = m\angle ABC \) and \( b = \angle EDF \), then \( 0 < a < b < 180 \) (By Axiom A1)
2. Set up a coordinate system for the rays from \( E \) on the \( D \)-side of line \( \overrightarrow{EF} \), with 0 assigned to the ray \( \overrightarrow{EF} \). (Protractor Postulate)
3. If the coordinate of ray \( \overrightarrow{ED} \) is \( x \), then \( b = m\angle EDF = |0 - x| = x \).
4. Let ray \( \overrightarrow{EG} \) be the unique ray having coordinate \( a \). Then \( \overrightarrow{Ed} - \overrightarrow{EG} - \overrightarrow{EF} \).
5. \( m\angle GEF = |a - 0| = a = m\angle ABC \)

### Perpendicularity

**Definition 3.39** If \( A - B - C \), then \( \overrightarrow{BC} \) and \( \overrightarrow{BA} \) are called opposite or opposing rays.

**Theorem 3.40** Given a ray \( \overrightarrow{BA} \), then its opposing ray \( \overrightarrow{BC} \) exists and is unique.
Proof: Need to show existence and uniqueness.
existence: (by construction) Let $\overrightarrow{BC} = \{X : X = B - A \text{ or } X = B\}$.
uniqueness: trivial.

Definition 3.41 Two angles are said to form a linear pair iff they have one side in common and the other two sides are opposite rays. We call any two angles whose angle measures sum to 180 a supplementary pair, or simply, supplementary, and two angles whose angle measures sum to 90, complementary.

Theorem 3.42 Two angles that are supplementary, or complementary, to the same angle have equal measures.

The proof is trivial.

Axiom 3.43 (A-4) A linear pair of angles is a supplementary pair.

Example 3.44 Certain rays on one side of line $\overrightarrow{BF}$ have their coordinates as indicated in Figure ??, and $m\angle GBF = 80$. Ray $\overrightarrow{BA}$ is opposite ray $\overrightarrow{BC}$, and $\overrightarrow{BG}$ is opposite $\overrightarrow{BE}$. Using the betweenness relations evident from the figure, find:
(a) $m\angle ABG$
(b) $m\angle DBG$
(c) The coordinate of ray $\overrightarrow{BE}$. 
Definition 3.45 A right angle is any angle having measure 90. Two (distinct) lines $\ell$ and $m$ are said to be perpendicular, and we write $\ell \perp m$, iff they contain the sides of a right angle. (For convenience, segments are perpendicular iff they lie, respectively, on perpendicular lines. Similar terminology applies to a segment and ray, two rays, and so on. )

Definition 3.46 acute angle: any angle whose measure is less than 90, and obtuse angle-any angle whose measure is greater than 90.

Lemma 3.47 If two lines are perpendicular, they form four right angles at their point of intersection.

Proof: Outline:
(1) If $\ell \perp m$, there are four ways in which $\ell$ and $m$ can contain the sides of a right angle.
(2) For each of the case, use the Linear Pair Axiom, to conclude that all angles are equal to 90.

Definition 3.48 adjacent angles: are two distinct angles with a common side and having no interior points in common.
Questions: Think about how many cases can we have for two angles with a common side. Which of the cases give two adjacent angles?

Theorem 3.49 If line $\overrightarrow{BD}$ meets segment $\overline{AC}$ at an interior point $B$ on that segment, then $\overrightarrow{BD} \perp \overline{AC}$ iff the adjacent angles at $B$ have equal measures.

Proof: Need to prove both forward and converse statements.

Theorem 3.50 Given a point $A$ on line $\ell$, there exists a unique line $m$ perpendicular to $\ell$ at $A$.

Proof: Need to show existence and uniqueness.
existence: construct a line $\overrightarrow{AB}$ such that the adjacent two angles at $A$ have the equal measure, then by Theorem 3.50, $\overrightarrow{AB} \perp \ell$.
uniqueness: Assume $\overrightarrow{AC} \perp \ell$, want to show $\overrightarrow{AC} = \overrightarrow{AB}$.

$m\angle DAB = m\angle DAC = 90$

Now use Protractor postulate, set up a coordinate system, such that the coordinate of $\overrightarrow{AB} = 0$, then by the protractor postulate, the coordinate of $\overrightarrow{AB}$ is 90, and so is $\overrightarrow{AC}$, so we must have $\overrightarrow{AC} = \overrightarrow{AB}$. One then similarly can show the other two halves of $\overrightarrow{AC}$ and $\overrightarrow{AB}$ are equal. That concludes $\overrightarrow{AC} = \overrightarrow{AB}$.

$\square$ Vertical angles
Definition 3.51 Vertical angles: are two angles having the sides of one opposite the sides of the other.

Theorem 3.52 (Vertical Pair Theorem) Vertical angles have equal measures.

Proof: Use the Linear Pair Axiom twice and Theorem 3.42.
Chapter 4

Foundations of Geometry 2

4.1 Triangles, Congruence Relations, SAS Hypothesis