Review Sheet

1. Basic Concepts

A **polynomial** is an expression in the form of

\[ a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \]

where \(a_0, \ldots, a_n\) are constant real numbers and \(x\) is the variable and \(n\) is a **positive** integer, called **degree** of the polynomial.

Example:

\[ f(x) = 3x^2 + x + 1 \]

while

\[ f(x) = x^2 + x^{1/2} + 3 \]

is **not** a polynomial.

A **rational expression** is an expression of the form

\[ R(x) = \frac{p(x)}{q(x)} \]

where \(p, q\) are polynomials in \(x\) and \(q \neq 0\). For example

\[ R(x) = \frac{2x^2 + 2}{3x} \]

Note

\[ \frac{x^2 + 1}{\sqrt{x}} \]

is **not** a rational function.

A **rational number** is a number which can be written into the form \(\frac{m}{n}\), where \(m, n\) are integers and \(n \neq 0\). For example:

- Integers: \(-1, 0, 1, \cdots\)
- Fractions: \(\frac{2}{3}\)
- Decimals with fixed digits: 2.3
- Decimals with repeated tail: \(0.\overline{333} = \frac{1}{3}\) and \(0.\overline{323232} = \frac{32}{99}\)

An **irrational number** is a real number which is not a rational number. For example: \(\sqrt{2}, \pi, e \approx 2.71828 \cdots, \ln 2, \cdots\)

The set of Rational numbers is a subset of the set of Real numbers.

\[ \mathbb{Q} \subset \mathbb{R}, \]

where \(\mathbb{Q}\) is the set of rational numbers, and \(\mathbb{R}\) is the set of real numbers.
A basic **complete square formula**

\[(a \pm b)^2 = a^2 \pm 2ab + b^2,\]

**Truncation and Rounding**

Truncating the number 23.9866 to two decimal points, the result is 23.98.
while rounding the number 23.9866 to two decimal points, the result is 23.99.

A function \(f\) is **even** if

\[f(-x) = f(x)\]

its graph is symmetric about \(y\)-axis, and \(f\) is **odd** if

\[f(-x) = -f(x)\]

its graph is symmetric about the origin.

For example: \(f(x) = x^2\) and \(f(x) = x^3\)

2. **Symmetry** Let \(P_1 = (1, -1)\) and \(P_2 = (x_2, y_2)\).

(a) If \(P_1\) and \(P_2\) are symmetric with respect to \(x\)-axis, then we must have \(x_2 = \ldots\),
and \(y_2 = \ldots\).

(b) If \(P_1\) and \(P_2\) are symmetric with respect to \(y\)-axis, then we must have \(x_2 = \ldots\),
and \(y_2 = \ldots\).

(c) If \(P_1\) and \(P_2\) are symmetric with respect to the origin, then we must have \(x_2 = \ldots\),
and \(y_2 = \ldots\).

3. **Roots of a polynomial** If \(a\) is a root of a polynomial \(p\) in \(x\), then \((x - a)\) is one of
the factor of \(p\), namely: we must have

\[p(x) = (x - a)q(x)\]

where \(q\) is a polynomial with degree 1 less than the degree of \(p\).

Example: Find a polynomial of degree 2 whose zeros are -1, 1.

4. **Distance and Midpoint formulae of two points**

Let \(P_1(x_1, y_1), P_2(x_2, y_2)\), then

\[d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\]

and if \(M\) is the midpoint, then

\[M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\]
5. **Absolute Value**

If

\[ |x| = a \quad \text{then} \quad x = \pm a \]

and

\[ |x| < a \quad \text{then} \quad -a < x < a \]

\[ |x| > a \quad \text{then} \quad x > a \quad \text{or} \quad x < -a \]

6. **Equations of lines**

- **Slope-intercept form:** \( y = mx + b \)
- **Point-slope Form:** \( y - y_1 = m(x - x_1) \)

If two lines are parallel to each other, then \( m_1 = m_2 \)

if Two lines are perpendicular, then \( m_1m_2 = -1 \) or \( m_1 = -\frac{1}{m_2} \) provided neither slope is zero.

7. **Equations of a circle**

**Standard form:**

\[
(x - a)^2 + (y - b)^2 = r^2
\]

**General Form:**

\[
x^2 + y^2 + ax + by + c = 0
\]

under some conditions.

**Question:** how to find the center and radius of a circle from its general form?

**Answers:** By completing the squares for \( x \) and \( y \) respectively.

Example:

\[
x^2 + y^2 - 4x + 6y - 12 = 0
\]

8. **Graphing techniques**

(1) Vertical Shifting \( f(x) \pm k \)

(2) Horizontal Shifting \( f(x \pm k) \)

(3) Vertical Scaling \( kf(x) \) and \( \frac{1}{k}f(x) \) if \( k > 1 \)

(4) Horizontal Scaling: \( f(kx) \) and \( f(\frac{1}{k}x) \) if \( k > 1 \)

(5) Reflection about \( y \)-axis: \( f(x) \) and \( f(-x) \)

(6) Reflection about \( x \)-axis: \( f(x) \) and \( -f(x) \)

Example: Find the function after the following transformation applied to the graph of \( f(x) = x^2 \).

i) Shift left 2 units.

ii) after step (i), shift the graph obtained in step (i) down 2 units.

iii) after step (ii), reflect the graph obtained in step (ii) about the \( y \)-axis.

9. **Functions**

**Definition:** A function is a mapping that for each element in the **domain**, there is one and only one value in the **range** corresponding to it.

Geometrically, we can use **Vertical line Test** to test if a mapping is a function.

**Domain:** the set of all elements such that the we can perform the function operation.
For example: \( f(x) = \frac{1}{x-3}, \ g(x) = \sqrt{x-3}. \)

**Range**: the set of all elements corresponding to all the elements in the domain of the function.

**1-1 Function**: A function is 1-1 if for different elements \( x \) in the domain there are different \( y \) values in the range corresponding to them. Geometrically, we can use **Horizontal Line Test** to test the 1-1 property of a function.

If the function is 1-1, then we can find its range by finding the domain of its inverse function. Example: For \( f(x) = \frac{x+1}{x-1}, \)

1) Find the inverse function \( f^{-1} \) of \( f; \)
2) Using the inverse function, find the range of \( f. \)

Otherwise we can use some other methods.

10. **Operations of Functions**

   Sum, Difference, Product, Quotient of two functions \( f \) and \( g \). Note the domain of \( f \pm g \) or \( fg \) is the intersection of the domains of \( f \) and \( g \), and the domain of \( \frac{f}{g} \) is the intersection of the domains of \( f \) and \( g \), also excluding the points where \( g \) is zero.

   For example: Let \( f(x) = \sqrt{x-3}, \) and \( g(x) = \frac{1}{1-x}. \)

   i) Find the domains of \( f \) and \( g \) respectively.
   ii) Find \( \left( \frac{f}{g} \right) \) and its domain

11. **Composition of functions**

\[
(f \circ g)(x) := f(g(x))
\]

For example: Let \( f(x) = x^2 - 1, \) and \( g(x) = 2x. \) Find \( (f \circ g)(1) \)

Find the domain of \( f \circ g \) if \( f(x) = \frac{1}{x-1} \) and \( g(x) = \frac{1}{x-2}. \)

In this case note for \( x \) in the domain of \( f \circ g, \)
1) \( x \) must be in the domain of \( g; \)
2) \( g(x) \) must be in the domain of \( f. \)

12. **Quadratic Functions** \( f(x) = ax^2 + bx + c \)

**Roots (x-intercepts)**:

**Determinant**: \( \Delta = b^2 - 4ac. \)

if \( \Delta > 0, \)
\[
x = \frac{-b \pm \sqrt{\Delta}}{2a}
\]

if \( \Delta = 0, \)
\[
x = \frac{-b}{2a}
\]

if \( \Delta < 0, \) No real solutions(no \( x \)-intercept).

**Vertex**: is at

\[
\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)
\]
axis of symmetry: the vertical line $x = -\frac{b}{2a}$.

**Direction of parabola:**

$a > 0$, parabola open upward, the function has a **minimum** at the vertex.

$a < 0$, parabola open downward, the function has a **maximum** at the vertex.

13. **Intercepts of the graph of a function** $f$

- **$x$-intercepts**: are those $x$ such that $f(x) = 0$;
- **$y$-intercept**: is $f(0)$.

For example: $R(x) = \frac{x(x-1)}{(x-2)(x+1)}$.

14. **Vertical and Horizontal Asymptotes of the graph of a rational function**

For a rational function $R(x) = \frac{p(x)}{q(x)}$, its

- **Vertical asymptotes are**: $x = r$, where $q(r) = 0$ but $p(r) \neq 0$
- **For Horizontal asymptotes**:
  1. if $R$ is proper, then $y = 0$ is the HA.
  2. if $\text{deg} p = \text{deg} q$, then $y = \text{ratio of the coefficients of the terms of highest power from top and bottom}$
  3. if $\text{deg} p = 1 + \text{deg} q$, then the graph has an **Oblique Asymptote**, perform the polynomial division.
  4. Otherwise:

\[
R(x) = s(x) + \frac{r(x)}{q(x)},
\]

where, $s$ is a polynomial with degree less than that of $p$, and $\text{degr} < \text{degq}$.

For example: $R(x) = \frac{x(x-1)}{(x-2)(x+1)}$.

15. **Solving inequalities involving polynomials or rational functions**

- $(a) \quad x^2 - 2x - 3 \leq 0$
- $(b) \quad \frac{(x+1)(x-1)}{(x-3)^2} > 0$

**Remark**: A polynomial function has no HA or VA.